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# THE MULTIPLICATIVE STRUCTURE OF $K(n)^*(BA_4)$

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## Abstract

Let  $K(n)^*(-)$  be a Morava  $K$ -theory at the prime 2. Invariant theory is used to identify  $K(n)^*(BA_4)$  as a summand of  $K(n)^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2)$ . Similarities with  $H^*(BA_4; \mathbb{Z}/2)$  are also discussed.

## Introduction

Let  $G$  be a finite group, and let  $N$  and  $C$  denote respectively the normalizer and the centralizer of a  $p$ -Sylow subgroup  $H$  of  $G$ .

For a large family of cohomology theories including the Brown-Peterson cohomology  $BP^*(-)$  and Morava  $K$ -theories  $K(n)^*(-)$ , the author described  $h^*(BG)$  when  $H$  is cyclic [3], and discussed the case “ $p$ -rank  $(H) < 3$ ” in [5], proving in particular that  $h^*(BG)$  is generated as  $h^*$ -module by at most two elements if  $|N : C|$  divides  $p - 1$ .

Results in this paper show that the condition above is really necessary, in fact we have

**Theorem 0.1.** *Let  $K(n)^*(-)$  be a Morava  $K$ -theory at the prime 2.  $K(n)^*(BA_4)$  restricts to those elements in*

$$K(n)^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2) \cong K(n)^*[x, y]/(x^{2^n}, y^{2^n})$$

which algebraically depend on

$$\begin{aligned}\bar{\sigma} &= x^2 + y^2 + xy + \nu_n(x^{2^{n-1}+1}y^{2^{n-1}} + x^{2^{n-1}}y^{2^{n-1}+1}), \\ \bar{\tau}_1 &= x^3 + y^3 + x^2y + \nu_n(x^{2^{n-1}}y^{2^{n-1}+2}), \\ \bar{\tau}_2 &= x^3 + y^3 + xy^2 + \nu_n(x^{2^{n-1}+2}y^{2^{n-1}}).\end{aligned}$$

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This paper has several motivations. The knowledge of  $K(n)^*(BA_4)$  could help to have explicit formulæ for the  $K(n)^*$ -Dickson classes. Furthermore, similarities among  $H^*(BA_4; \mathbb{Z}/2)$  and  $K(n)^*(BA_4)$  suggest to study  $K(n)^*(BA_m)$  —whose rank as  $K(n)^*$ -module can be calculated [6]— to get information on  $H^*(BA_m; \mathbb{Z}/2)$  which is not entirely known for  $m \geq 16$  (see [1] for the cohomology of several alternating groups).

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### 1. Preliminaries. $H^*(BA_4)$

From now on  $V$  will denote the group  $\mathbb{Z}/2 \times \mathbb{Z}/2$ , and  $H^*(-)$  ordinary cohomology with coefficients in  $\mathbb{Z}/2$ .

In [7], the authors describe  $H^*(PSL_2\mathbb{F}_q)$  for any odd  $q$ : they first calculate the cohomology of the generalized quaternion group  $Q_{2^{n+1}}$  of order  $2^{n+1}$ , and then use the diagram

$$\begin{array}{ccccc} \mathbb{Z}/2 & \longrightarrow & SL_2\mathbb{F}_q & \longrightarrow & PSL_2\mathbb{F}_q \\ \parallel & & \uparrow i & & \uparrow j \\ \mathbb{Z}/2 & \longrightarrow & Q_{2^{n+1}} & \longrightarrow & D_n \end{array}$$

where  $D_n$  is the dihedral group of order  $2^n$ , rows are fibrations, and  $i$  and  $j$  are inclusions of 2-Sylow subgroups. Nevertheless, we show in this section that the special case

$$PSL_2\mathbb{F}_3 \cong A_4$$

can be approached in a more direct way.

The alternating group  $A_4$  is the central term of the short exact sequence of groups

$$0 \longrightarrow V \longrightarrow A_4 \longrightarrow \mathbb{Z}/3 \longrightarrow 0,$$

therefore for any mod 2 cohomology theory  $h^*(-)$ ,  $h^*(BA_4)$  is isomorphic to the ring of invariants  $[h^*(BV)]^{\mathbb{Z}/3}$  under the action determined by the map  $h^*(B\phi)$  induced by an automorphism  $\phi$  of order 3 in  $\text{Aut}(V)$ . On  $H^*(BV) \cong \mathbb{F}_2[x, y]$  the action of a generator of  $\mathbb{Z}/3 \leq GL(V)$  is

$$x \xrightarrow{\alpha_H} y \quad \text{and} \quad y \xrightarrow{\alpha_H} x + y.$$

Consider now the map  $\Phi$  from  $F_2[x, y]$  to itself which maps any element  $c$  to the sum

$$\Phi(c) = c + \alpha_H(c) + \alpha_H^2(c);$$

$\Phi$  is commonly known as norm map. It is easy to see that

$$\mathrm{Im} \Phi = [H^*(BV)]^{\mathbb{Z}/3};$$

furthermore the image of  $\Phi$  restricted to the set of monomials generates  $[H^*(BV)]^{\mathbb{Z}/3}$  regarded as graded  $\mathbb{F}_2$ -vector space.

The invariant of lowest positive degree in  $\mathbb{F}_2[x, y]$  is

$$\sigma = \Phi(xy) = x^2 + xy + y^2.$$

This element is actually the Dickson class known in literature as  $Q_{2,1}$  (see [9]).

The reader will find the relevant invariant theoretic computation in [2] to prove the algebraic dependence of every invariant on  $\Phi(xy)$ ,  $\Phi(x^2y)$ ,  $\Phi(xy^2)$ . In fact we have the following proposition.

**Proposition 1.1.** *As a graded ring,  $H^*(BA_4)$  is isomorphic to*

$$\mathbb{F}_2[\sigma, \tau_1, \tau_2]/R,$$

where  $\deg \sigma = 2$ ,  $\deg \tau_1 = \deg \tau_2 = 3$ , and  $R$  is the ideal generated by

$$\sigma^3 + \tau_1^2 + \tau_1\tau_2 + \tau_2^2.$$

The proposition above can be restated in terms of pure invariant theory.

**Corollary 1.2.** *Suppose that a  $\mathbb{Z}/3$ -action on  $\mathbb{F}_2[x, y]$  is given by*

$$x \longrightarrow y \quad \text{and} \quad y \longrightarrow x + y.$$

*The ring of the invariants is a polynomial ring generated by*

$$\sigma = x^2 + y^2 + xy, \quad \tau_1 = x^3 + y^3 + x^2y, \quad \tau_2 = x^3 + y^3 + xy^2,$$

*quotiented by*

$$R = (\sigma^3 + \tau_1^2 + \tau_1\tau_2 + \tau_2^2).$$

## 2. The Morava $K$ -theory of $BA_4$

We recall that Morava  $K$ -theory at the prime 2 is a complex oriented cohomology theory with coefficients

$$K(n)^*(\{pt\}) = \mathbb{F}_2[\nu_n, \nu_n^{-1}]$$

where  $\deg \nu_n = -2(2^n - 1)$ , and we have

$$K(n)^*(BV) \cong K(n)^*[x, y]/(x^{2^n}, y^{2^n})$$

where  $\deg x = \deg y = 2$ . As noticed in section 1,  $K(n)^*(BA_4)$  is isomorphic to

$$[K(n)^*(BV)]^{\mathbb{Z}/3}$$

where the  $\mathbb{Z}/3$ -module structure is defined by the map  $K(n)^*(B\phi)$ , being  $\phi$  a generator of  $\mathbb{Z}/3 \leq \text{Aut}(V)$ . The following lemma helps to give a concrete description of the  $K(n)$ -invariants.

**Lemma 2.1.** *One of the two generators  $\phi$  of  $\mathbb{Z}/3 \leq \text{Aut}(V)$  acts as follows on  $K(n)^*(BV)$ :*

$$\alpha_K \stackrel{\text{def}}{=} K(n)^*(B\phi) : x \longrightarrow y \quad \text{and} \quad \alpha_K : y \longrightarrow x + y + \nu_n x^{2^{n-1}} y^{2^{n-1}}.$$

*Proof:* See [4]. ■

The element  $\alpha_K(y)$  is actually the formal sum of  $x$  and  $y$  with respect to the formal group law of mod 2 Morava  $K$ -theory

$$F_{K(n)}(x, y) \mod (x^{2^n}, y^{2^n}).$$

Consider now the norm map  $\Psi$  defined as follows:

$$\Psi : c \in K(n)^*(BV) \longmapsto c + \alpha_K(c) + \alpha_K^2(c) \in [K(n)^*(BV)]^{\mathbb{Z}/3}.$$

The map  $\Psi$  is obviously the analogue of  $\Phi$  defined in section 1: it is surjective, and the invariants regarded as  $\mathbb{F}_2$ -vector space are spanned by the image of  $\Psi$  restricted to monomials.

Notice also that we can equip

$$K(n)^*(BV) \cong K(n)^*[x, y]/(x^{2^n}, y^{2^n})$$

with a different  $\mathbb{Z}/3$ -module structure just by posing

$$\alpha_H(x) = y \quad \text{and} \quad \alpha_H(y) = x + y.$$

Abusing notation, we shall use again  $\Phi$  to denote the endomorphism defined on the generic element of  $K(n)^*(BV)$  as follows:

$$c \longmapsto c + \alpha_H(c) + \alpha_H^2(c).$$

We are ready now to prove our main result.

**Theorem 2.2.**  $K(n)^*(BA_4)$  restricts to those elements in  $K(n)^*(BV)$  which algebraically depend on

$$\Psi(xy) = \bar{\sigma}, \quad \Psi(x^2y) = \bar{\tau}_1 \quad \text{and} \quad \Psi(xy^2) = \bar{\tau}_2.$$

*Proof:* Since  $K(n)^*(-)$  is  $2(2^n - 1)$ -periodic we can look at classes in  $K(n)^*(BV)$  whose degree is between 2 and  $2(2^n - 1)$ . In this range, elements of type

$$\nu_n^2 x^h y^k$$

are necessarily zero, since either  $h$  or  $k$  is greater than  $2^n$ . It follows that for any monomial  $x^h y^k \in K(n)^*(BV)$  we have

$$(1) \quad \Psi(\nu_n x^h y^k) = \nu_n \Phi(x^h y^k).$$

An element  $c \in K(n)^*(BV)$  is invariant under  $\alpha_K$  if and only if  $\Psi(c) = c$ , and supposing

$$2 \leq t \leq 2(2^n - 1),$$

we have

$$c = p(x, y) + \nu_n q(x, y),$$

where  $p(x, y)$  and  $q(x, y)$  are homogeneous polynomials of  $\mathbb{F}_2[x, y]$  of degree  $t$  and  $t + 2(2^n - 1)$  respectively. If  $\Psi(c) = c$ , it follows from the considerations above that  $\Phi(p(x, y)) = p(x, y)$ , and by Corollary 1.2 there exists a polynomial  $r_1$  in three indeterminates such that

$$r_1(\sigma, \tau_1, \tau_2) = p(x, y).$$

Define now

$$\Psi(xy) = \bar{\sigma}, \quad \Psi(x^2y) = \bar{\tau}_1 \quad \text{and} \quad \Psi(xy^2) = \bar{\tau}_2.$$

The element

$$c - r_1(\bar{\sigma}, \bar{\tau}_1, \bar{\tau}_2) = \nu_n s(x, y)$$

is invariant under  $\alpha_K$ . Notice now that  $s(x, y)$  can be regarded as a polynomial in  $\mathbb{F}_2[x, y]$ ; it follows by (1) that  $s(x, y)$  is invariant under  $\alpha_H$ , and again by Corollary 1.2 there exists a polynomial  $r_2$  in three indeterminates such that

$$r_2(\sigma, \tau_1, \tau_2) = s(x, y).$$

We finally get

$$c = r_1(\bar{\sigma}, \bar{\tau}_1, \bar{\tau}_2) - \nu_n r_2(\bar{\sigma}, \bar{\tau}_1, \bar{\tau}_2)$$

as we claimed. ■

Theorem 2.2 also gives some information on  $K(n)^*(BA_5)$ . Notice in fact that 2-Sylow subgroups in  $A_5$  are abelian, and a 2-Sylow normalizer in  $A_5$  is isomorphic to  $A_4$ . It follows by a theorem in [8] that  $BA_4$  and  $BA_5$  are stably 2-homotopy equivalent. Hence the map induced by inclusion

$$K(n)^*(BA_5) \longrightarrow K(n)^*(BA_4)$$

is an isomorphism.

**Remark 2.3.** The element

$$\bar{\sigma}^3 + \bar{\tau}_1^2 + \bar{\tau}_1 \bar{\tau}_2 + \bar{\tau}_2^2$$

is zero in  $K(n)^*(BA_4)$ , as the analogous algebraic expression in  $\sigma, \tau_1, \tau_2$  for ordinary cohomology. The relation above is not however of minimal positive degree: the element

$$\nu_n^2 \bar{\sigma}^{2^n}$$

is zero and has degree four.

It is known that the subring of  $H^{\text{even}}(BA_4)$  generated by Chern classes is proper (see, for example [10, p. 100]), and the reader could ask if  $\bar{\sigma}, \bar{\tau}_1, \bar{\tau}_2$  are  $K(n)$ -Chern classes of suitable representations.

We recall that up to equivalence the group  $A_4$  has just four distinct complex irreducible representations. Three of them are one-dimensional, and their restriction to  $V$  is trivial. The fourth one has instead non-trivial total Chern class in  $K(n)^*(BA_4)$ , as the next proposition shows.

**Proposition 2.4.** *Let  $\xi$  be a 3-dimensional irreducible representation of  $A_4$ . The restriction  $\xi|_V$  to the 2-Sylow subgroup  $V$  has Chern classes*

$$c_1(\xi|_V) = \nu_n \bar{\sigma}^{2^{n-1}}, \quad c_2(\xi|_V) = \bar{\sigma}, \quad c_3(\xi|_V) = \bar{\tau}_1 + \bar{\tau}_2 + \nu_n \bar{\sigma}^{2^{n-1}+1}$$

in  $K(n)^*(BV)$ .

*Proof:* Let  $g_1$  and  $g_2$  be two generators in  $V$ . Consider two one-dimensional representations  $\rho_1$  and  $\rho_2$  defined as follows

$$\rho_i : g_i \longmapsto -1 \quad \rho_i : g_{3-i} \longmapsto 1$$

for  $i = 1, 2$ . The transfer  $\xi$  of  $\rho_1$  to  $A_4$  represents the equivalence class of the 3-dimensional irreducible representations of  $A_4$ ; its restriction to  $V$  is given by

$$\rho_1 \oplus \rho_2 \oplus (\rho_1 \otimes \rho_2).$$

It follows that the total Chern class  $c(\xi|_V)$  is equal to

$$(1+x)(1+y)(1+x+y+\nu_n x^{2^{n-1}} y^{2^{n-1}}).$$

Hence the proposition follows. ■

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